Dichotomy Result on 3-Regular Bipartite Non-negative Functions

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Counting Problem

Given a Boolean formula

Decision: is there a satisfying assignment ϕ?

Counting: how many satisfying assignments ϕ are there?

Holant Framework

Trace back to Valiant's holographic transformation [Val06]

Capture many counting problems in a natural way, e.g. counting perfect matchings

Provably more expressive than #CSP [FLS07]

Long line of research showing dichotomies in Holant framework [KC16, CGW16, Bac18]

Holant problem

Input: any *signature grid* $\Omega = (G, \mathcal{F}, \pi)$ where $G = (V, E)$ is a graph, $\mathcal{F}$ is a set of functions $[q]^k \to \mathbb{C}$, and π is a mapping from the vertex set V to $\mathcal{F}$.

Output: the *Holant value* $\text{Holant}_\Omega = \sum_\sigma \prod_{v \in V} f_v(\sigma|_{E(v)})$ where σ is a mapping $E \to [q]$, $f_v(\cdot) := \pi(v) \in \mathcal{F}$, and $E(v)$ denotes the set of incident edges of v.

Holant problem; example

Example

Let $q = \{0, 1\}$ and $\mathcal{F} = \{\textsc{At-Most-One}_k\}$, then Holant_Ω counts the number of matchings.

Example

Let $q = \{0, 1, \ldots, k-1\}$ and $\mathcal{F} = \{\textsc{All-Distinct}_k\}$, then Holant_Ω counts the number of proper edge colorings using at most k colors.

Dichotomy results in Holant framework

Dichotomy says a problem is *either* tractable *or* #P-complete, despite Ladner's theorem (counting version).

See Guo and Lu's survey *On the Complexity of Holant Problems*; Cai and Chen's book *Complexity Dichotomies for Counting Problems*; Shuai's thesis *Complexity Classification of Counting Problems on Boolean Variables* for more information.

Dichotomy results in Holant framework; example

Theorem (Cai, Guo, Williams; 2012)

A Holant problem over an arbitrary set of complex-valued symmetric constraint functions $\mathcal{F}$ on Boolean variables is $\#$P-complete unless:

- *every function in $\mathcal{F}$ has arity at most two;*
- *$\mathcal{F}$ is transformable to an affine type;*
- *$\mathcal{F}$ is transformable to a product type;*
- *$\mathcal{F}$ is vanishing, combined with the right type of binary functions;*
- *$\mathcal{F}$ belongs to a special category of vanishing type Fibonacci gates.*

in which the Holant value can be computed in polynomial time.

Bipartite Holant problem

Restrict underlying graph to be bipartite.

Input: any *signature grid* $\Omega = (G, \mathcal{F}, \mathcal{G}, \pi)$ where $G = (V, U, E)$ is a bipartite graph, $\mathcal{F}$ and $\mathcal{G}$ are two sets of functions $[q]^k \to \mathbb{C}$, and π is a mapping from the vertex set $V \cup U$ to $\mathcal{F} \cup \mathcal{G}$ such that $\pi(V) \subseteq \mathcal{F}$ and $\pi(U) \subseteq \mathcal{G}$.

Output: $\mathrm{Holant}_\Omega = \sum_{\sigma} \prod_{v \in V} f_v(\sigma|_{E(v)}) \prod_{u \in U} g_u(\sigma|_{E(u)})$ where σ is a mapping $E \to [q]$, $f_v(\cdot) := \pi(v) \in \mathcal{F}$, $g_u(\cdot) := \pi(u) \in \mathcal{G}$, and $E(v)$ denotes the set of incident edges of v.

#CSP

Fix a domain $D = \{1, 2, \ldots, d\}$ and a set of complex-valued functions $\mathcal{F} = \{f_1, f_2, \ldots, f_h\}$ where $f_i : D^{r_i} \to \mathbb{C}$ for some r_i.

Input: a tuple $\mathrm{x} = (x_1, \ldots, x_n)$ of variables over D and a collection I of tuples $(f, i_1, \ldots, i_r)$ in which f is an r-ary function from $\mathcal{F}$ and $i_1, \ldots, i_r \in [n]$.

Output: the *partition function* $Z(I) := \sum_{\mathrm{x} \in D^n} F_I(\mathrm{x})$ where $F_I(\mathrm{x}) := \prod_{(f, i_1, \ldots, i_r) \in I} f\left(x_{i_1}, \ldots, x_{i_r}\right)$

Observe this is the bipartite Holant problem with $\mathcal{F}$ on one side and $\textsc{Equality} := \{=_k \text{ for all } k \in \mathbb{N}\}$ on the other side!

Our main result

We initiate the study of Holant problems on bipartite graphs.

Specifically, We prove a dichotomy result on a class of 3-regular bipartite graph Holant problem, namely Holant($f \mid =_3$) where $f = [x_0, x_1, x_2, x_3]$ is an arbitrary non-negative weighted symmetric Boolean constraint function and $=_3$ is the $\textsc{Equality}_3$ function.

This is the most basic yet non-trivial bipartite setting.

A mere starting point for understanding bipartite Holant problems: almost every generalization is an open problem at this point,

- including more than one constraint function on either side;
- other regularity parameter r;
- real or complex valued constraint functions which allow cancellations;
- ...

Hard problems become easy for bipartite graph?

Theorem (König)

In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.

Easy to find the minimum cardinality vertex cover in bipartite graphs, which is NP-hard for general graphs. Is there new algorithm in bipartite counting?

Remark

sometimes counting can be hard even when its "underlying" decision problem is easy, e.g. counting the perfect matchings in a bipartite graph is #P-complete [Val79].

Main theorem

Theorem (F. & Cai)

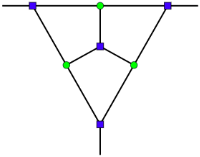
Holant$\{[x_0, x_1, x_2, x_3] | (=_3)\}$ *where* $x_i \geq 0$ *for* $i = 0, 1, 2, 3$ *is #P-hard except in the following cases, for which the problem is in* FP*.*

1. $[x_0, x_1, x_2, x_3]$ *is degenerate;*
2. $x_1 = x_2 = 0$*;*
3. $[(x_1 = x_3 = 0) \wedge (x_0 = x_2)]$ *or* $[(x_0 = x_2 = 0) \wedge (x_1 = x_3)]$*.*

Condition 2 is called *general equality* and condition 3 is a special case of what is called *affine class* [CC17]; both are within the tractable cases of #CSP.

Gadget

Proofs of previous dichotomies make substantial use of *gadgets*



(a) G_1

Definition

A bipartite gadget is a bipartite graph $G = (U, V, E_{\mathrm{in}}, E_{\mathrm{out}})$ with internal edges E_{in} and dangling edges E_{out}. Suppose there are m and n dangling edges internally incident to vertices from U and V. These $m + n$ dangling edges correspond to Boolean variables $x_1, \ldots, x_m, y_1, \ldots, y_n$ and the gadget defines a function

$$f(x_1, \ldots, x_m, y_1, \ldots, y_n) = \sum_{\sigma : E_{\mathrm{in}} \to \{0,1\}} \prod_{u \in U} f\left(\widehat{\sigma}|_{E(u)}\right) \prod_{v \in V} (=_3)\left(\widehat{\sigma}|_{E(v)}\right),$$

where $\widehat{\sigma}$ denotes the extension of σ by the assignment on the dangling edges.

Main obstacle

When the graph is bipartite and r-regular, there is a number theoretic limitation as to what types of gadgets one can possibly construct.

Every constructible gadget must have a rigid arity restriction; e.g., if the gadget represents a constraint function that can be used for a vertices in U or in V, the arity (the number of input variables) of the function must be congruent to 0 modulo r.

In particular, one cannot form self-loop since that will break the bipartite structure, which is one of the heavily used techniques in proving previous dichotomy result.

New technique

Interpolate degenerate straddled bipartite function and split them into unary functions.

By connecting one unary function with a ternary function, we get a binary function and thus reduces to the following scenario:

Theorem (Kowalczyk & Cai; 2016)

Suppose $a, b \in \mathbb{C}$*, and let* $X = ab$*,* $Z = \left(\frac{a^3+b^3}{2}\right)^2$*. Then* $\mathrm{Holant}([a, 1, b] \mid (=_3))$ *is #P-hard except in the following cases, for which the problem is in* P*.*

1. $X = 1$;
2. $X = Z = 0$;
3. $X = -1$ *and* $Z = 0$;
4. $X = -1$ *and* $Z = -1$.

New technique; splitting

Lemma

Let f and g be two non-negative valued signatures. If a degenerate nonnegative binary straddled signature $\left(\begin{smallmatrix} 1 & x \\ y & xy \end{smallmatrix}\right)$ can be obtained or interpolated in the problem $\text{Holant}(f|g)$, *then*

$$\text{Holant}(f|\{g,[1,x]\}) \leq_T \text{Holant}(f|g).$$

A similar statement holds for adding the unary $[1,y]$ on the LHS.

Proof sketch.

Given any bipartite signature gird $\Omega = (G,\pi)$ for problem $\text{Holant}(f|\{g,[1,x]\})$, we replace every $[1,x]$ by $\left(\begin{smallmatrix} 1 & x \\ y & xy \end{smallmatrix}\right)$ and "use up" the extra $[1,y]$'s by connecting them to f. This will compute $(\text{Holant}(G))^s$ for some s determined by the arities of f and g and introduce a global factor, which doesn't matter for the sake of complexity. $\square$

New technique; interpolation

Lemma

Given the binary straddled signature $G_1 = \begin{pmatrix} 1 & x_2 \\ x_1 & x_3 \end{pmatrix}$ *with* $x_1 \neq 0$ *and* $\Delta := \sqrt{(1-x_3)^2 + 4x_1x_2} > 0$, *we can get unary signatures* $[1, x]$ *on RHS or* $[y, 1]$ *on LHS where* $x = \frac{\Delta-(1-x_3)}{2x_1}$ *and* $y = \frac{\Delta+(1-x_3)}{2x_1}$.

Proof.

The Jordan Canonical Form for $G_1 = \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix}^{-1}$ and $D = \frac{1}{x+y} \begin{pmatrix} y & xy \\ 1 & x \end{pmatrix} = \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x & y \\ 1 & 1 \end{pmatrix}^{-1}$. Given any signature grid Ω where the binary degenerate straddled signature D appears n times, we form gadgets G_1^s where $0 \leq s \leq n$ by iterating the G_1 gadget s times and replacing each occurrence of D with G_1^s. Denote the resulting signature grid as Ω_s.

New technique; interpolation continued

Proof (Cont.)

We stratify the assignments in the Holant sum for Ω according to assignments to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ as:

1. $(0, 0)$ i times;
2. $(1, 1)$ j times;

with $i + j = n$. Let $c_{i,j}$ be the sum over all such assignments of the products of evaluations (including the contributions from $\left(\begin{smallmatrix} -x & y \\ 1 & 1 \end{smallmatrix}\right)$ and its inverse). Then we have

$$\text{Holant}_{\Omega_s} = \sum_{i+j=n} \left(\lambda^i \mu^j\right)^s \cdot c_{i,j}$$

and $\text{Holant}_{\Omega} = c_{0,n}$. Since $\Delta > 0$, the coefficients form a full rank Vandermonde matrix. Thus we can interpolate D by solving the linear system of equations in polynomial time. Ignoring a nonzero factor, we may split D into unary signatures $[y, 1]$ on LHS or $[1, x]$ on RHS. $\square$

Holographic transformation

Theorem (Valiant; 2008)

Let $\mathcal{F}$ and $\mathcal{G}$ be sets of complex-valued signatures over a domain of size q. Suppose Ω is a bipartite signature grid over $(\mathcal{F} \mid \mathcal{G})$. If $T \in \mathsf{GL}_q(\mathbb{C})$, then

$$\textit{Holant}_q(\Omega; \mathcal{F} \mid \mathcal{G}) = \textit{Holant}_q\left(\Omega'; \mathcal{F}T \mid T^{-1}\mathcal{G}\right)$$

where Ω' is the corresponding signature grid over $\left(\mathcal{F}T \mid T^{-1}\mathcal{G}\right)$.

Pl-#HyperGragh-Moderate-3-Cover

Input: A planar 3-uniform 3-regular hypergraph G.
Output: The number of subsets of hyperedges that cover every vertex with no vertex covered three times.

This is exactly the problem Pl-Holant$([0,1,1,0] | [1,0,0,1])$. By performing holographic reduction by Hadamard matrix $H = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, this is the same as Pl-Holant$([2,0,2,0] | [3,0,-1,0])$ since

$$H^{\otimes 3}(=_3) = H^{\otimes 3}\left[\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^{\otimes 3} + \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^{\otimes 3}\right] = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)^{\otimes 3} + \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)^{\otimes 3} = [2,0,2,0]$$

and

$$\begin{aligned}[0,1,1,0](H^{-1})^{\otimes 3} &= \frac{1}{8}\left[(1,\ \ 1)^{\otimes 3} - (1,\ \ 0)^{\otimes 3} - (0,\ \ 1)^{\otimes 3}\right] H^{\otimes 3} \\ &= \frac{1}{4}[3,0,-1,0].\end{aligned}$$

which can be computed efficiently by matchgates [CC17].

Future work

Extend the technique in this work along with other tricks, a dichotomy for $\text{Holant}([x_0, x_1, x_2, x_3] \mid =_3)$ when $x_i \in \mathbb{Q}$ has recently been proven.

Immediate questions:

- Real or even complex-valued functions?
- $\mathcal{F}$ has more than one function?
- Drop the $=_3$ assumption?
- ...

Thank you!

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