Bipartite 3-Regular Counting Problems with Mixed Signs

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Counting Problem

Given a Boolean formula

Decision: is there a satisfying assignment ϕ?

Counting: how many satisfying assignments ϕ are there?

Holant Framework

Trace back to Valiant's holographic transformation [Val06]

Capture many counting problems in a natural way, e.g. counting perfect matchings

Provably more expressive than #CSP [FLS07]

Long line of research showing dichotomies in Holant framework [KC16, CGW16, Bac18]

Holant problem

Input: any *signature grid* $\Omega = (G, \mathcal{F}, \pi)$ where $G = (V, E)$ is a graph, $\mathcal{F}$ is a set of functions $[q]^k \to \mathbb{C}$, and π is a mapping from the vertex set V to $\mathcal{F}$.

Output: the *Holant value* $\text{Holant}_\Omega = \sum_\sigma \prod_{v \in V} f_v(\sigma|_{E(v)})$ where σ is a mapping $E \to [q]$, $f_v(\cdot) := \pi(v) \in \mathcal{F}$, and $E(v)$ denotes the set of incident edges of v.

Holant problem; example

Example

Let $q = \{0, 1\}$ and $\mathcal{F} = \{\textsc{At-Most-One}_k\}$, then Holant_Ω counts the number of matchings.

Example

Let $q = \{0, 1, \ldots, k-1\}$ and $\mathcal{F} = \{\textsc{All-Distinct}_k\}$, then Holant_Ω counts the number of proper edge colorings using at most k colors.

Dichotomy results in Holant framework

Dichotomy says a problem is *either* tractable *or* #P-complete, despite Ladner's theorem (counting version).

See Guo and Lu's survey *On the Complexity of Holant Problems*; Cai and Chen's book *Complexity Dichotomies for Counting Problems*; Shuai's thesis *Complexity Classification of Counting Problems on Boolean Variables* for more information.

Dichotomy results in Holant framework; example

Theorem (Cai, Guo, Williams; 2012)

A Holant problem over an arbitrary set of complex-valued symmetric constraint functions $\mathcal{F}$ on Boolean variables is #P-complete unless:

- *every function in $\mathcal{F}$ has arity at most two;*
- *$\mathcal{F}$ is transformable to an affine type;*
- *$\mathcal{F}$ is transformable to a product type;*
- *$\mathcal{F}$ is vanishing, combined with the right type of binary functions;*
- *$\mathcal{F}$ belongs to a special category of vanishing type Fibonacci gates.*

in which the Holant value can be computed in polynomial time.

Bipartite Holant problem

Restrict underlying graph to be bipartite.

Input: any *signature grid* $\Omega = (G, \mathcal{F}, \mathcal{G}, \pi)$ where $G = (V, U, E)$ is a bipartite graph, $\mathcal{F}$ and $\mathcal{G}$ are two sets of functions $[q]^k \to \mathbb{C}$, and π is a mapping from the vertex set $V \cup U$ to $\mathcal{F} \cup \mathcal{G}$ such that $\pi(V) \subseteq \mathcal{F}$ and $\pi(U) \subseteq \mathcal{G}$.

Output: $\text{Holant}_\Omega = \sum_{\sigma} \prod_{v \in V} f_v(\sigma|_{E(v)}) \prod_{u \in U} g_u(\sigma|_{E(u)})$ where σ is a mapping $E \to [q]$, $f_v(\cdot) := \pi(v) \in \mathcal{F}$, $g_u(\cdot) := \pi(u) \in \mathcal{G}$, and $E(v)$ denotes the set of incident edges of v.

#CSP

Fix a domain $D = \{1, 2, \ldots, d\}$ and a set of complex-valued functions $\mathcal{F} = \{f_1, f_2, \ldots, f_h\}$ where $f_i : D^{r_i} \to \mathbb{C}$ for some r_i.

Input: a tuple $\mathrm{x} = (x_1, \ldots, x_n)$ of variables over D and a collection I of tuples $(f, i_1, \ldots, i_r)$ in which f is an r-ary function from $\mathcal{F}$ and $i_1, \ldots, i_r \in [n]$.

Output: the *partition function* $Z(I) := \sum_{\mathrm{x} \in D^n} F_I(\mathrm{x})$ where $F_I(\mathrm{x}) := \prod_{(f, i_1, \ldots, i_r) \in I} f(x_{i_1}, \ldots, x_{i_r})$

Observe this is the bipartite Holant problem with $\mathcal{F}$ on one side and $\textsc{Equality} := \{=_k \text{ for all } k \in \mathbb{N}\}$ on the other side!

Our main result

We initiate the study of Holant problems on bipartite graphs.

Specifically, We prove a dichotomy result on a class of 3-regular bipartite graph Holant problem, namely $\text{Holant}(f \mid =_3)$ where f is an arbitrary rational symmetric Boolean constraint function and $=_3$ is the $\textsc{Equality}_3$ function.

This is the most basic yet non-trivial bipartite setting and our result is a mere starting point for understanding bipartite Holant problems. Almost every generalization is an open problem at this point.

Theorem (Cai, F. & Liu)

The problem $\text{Holant}\{[f_0, f_1, f_2, f_3] \mid (=_3)\}$ *with* $f_i \in \mathbb{Q}$ $(i = 0, 1, 2, 3)$ *is #P-hard unless the signature* $[f_0, f_1, f_2, f_3]$ *is degenerate, Gen-Eq or belongs to the affine class.*

New phenomenon

We discover a set $\mathcal{F}$ with the property that for every $f \in \mathcal{F}$ the problem $\operatorname{Holant}(f \mid =_3)$ is planar P-time computable but #P-hard in general, yet its planar tractability is by a *combination* of a holographic transformation by Hadamard matrix $H = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ to FKT *together* with an independent global argument.

Lemma (Cai, F. & Liu)

The problem $\operatorname{Holant}([3a+b, -a-b, -a+b, 3a-b] \mid =_3)$ *is computable in polynomial time on planar graphs for all* a, b*, but is #P-hard on general graphs for all* $a \neq 0$.

Why remarkable?

Proof.

The following equivalence is by holographic transformation:

$$
\begin{aligned}
\text{Holant}\,(f \mid (=_3)) &\equiv_T \text{Holant}\left(fH^{\otimes 3} \mid (H^{-1})^{\otimes 3}(=_3)\right)\\
&\equiv_T \text{Holant}\,([0,0,a,b] \mid [1,0,1,0])\\
&\equiv_T \text{Holant}\,([0,0,a,0] \mid [0,0,1,0])\\
&\equiv_T \text{Holant}\,([0,a,0,0] \mid [0,1,0,0])
\end{aligned}
$$

where the last line is (up to a global nonzero factor) the perfect matching problem on 3-regular bipartite graphs. This problem is computable in polynomial time on planar graphs and #P-hard on general graphs. □

Why remarkable?

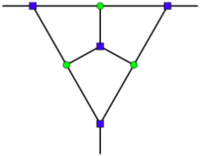
For counting CSP problems over Boolean variables, all problems that are #P-hard in general but P-time tractable on planar graphs are tractable by the following universal algorithmic strategy: a holographic transformation to matchgates followed by the FKT algorithm [CLX10].

On the other hand, for (non-bipartite) Holant problems with arbitrary symmetric signature sets, this category of problems (planar tractable but #P-hard in general) is completely characterized by two types [CFGW15] : (1) holographic transformations to matchgates, and (2) a separate kind that depends on the existence of "a wheel structure" (unrelated to holographic transformations and matchgates).

Here we have found the first instance where a new type has emerged!

Main obstacle I

When the graph is bipartite and r-regular, there is a number theoretic limitation as to what types of gadgets one can possibly construct.



Main obstacle II

```
mat = {{1, a^2, a, a*b}, {a, a*b, a^2, b^2}, {a^2, b^2, a*b, b*c}, {a*b, b*c, b^2, c^2}}
Vet = {{1}, {a}, {b}, {c}}

JordanDecomposition[mat.mat.Vet]
```

Out[*]=

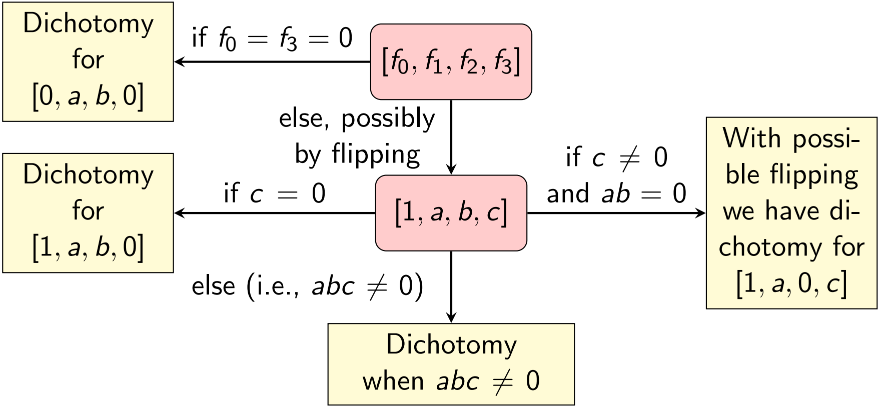
$$
\begin{aligned}
&\{\{\{-((-1-3a^3-a^4b-a^2b^2+ab^4+a^2b^2c+3b^3c^2+c^5+\\
&\quad\sqrt{}(1+10a^3+21a^6+8a^9+18a^4b+30a^7b+6a^2b^2+46a^5b^2+17a^8b^2+28a^3b^3+70a^6b^3-2ab^4+51a^4b^4+8a^2b^5+\\
&\qquad 66a^5b^5+22a^3b^6+4ab^7+17a^2b^8+8a^4bc+12a^7bc+2a^2b^2c+18a^5b^2c+20a^3b^3c+22a^6b^3c+8ab^4c+\\
&\qquad 66a^4b^4c+28a^2b^5c+70a^3b^6c+12ab^7c+8b^9c+4a^4bc^2+4a^7bc^2+8a^2b^2c^2+28a^5b^2c^2-6b^3c^2+6a^3b^3c^2+\\
&\qquad 51a^4b^4c^2+18a^2b^5c^2+30ab^7c^2+4a^2b^2c^3+8a^5b^2c^3+20a^3b^3c^3+4ab^4c^3+46a^2b^5c^3+4abc^4+8a^4bc^4+\\
&\qquad 8a^2b^2c^4+28a^3b^3c^4+8ab^4c^4+21b^6c^4-2c^5-6a^3c^5-2a^4bc^5+2a^2b^2c^5+18ab^4c^5+6a^2b^2c^6+10b^3c^7+c^{10}))/\\
&\quad(2(a+2a^4+2a^2b+4a^3b^2+ab^3+b^5+a^2bc+2ab^3c+a^2bc^2+b^2c^3))),\\
&\;-((-1-3a^3-a^4b-a^2b^2+ab^4+a^2b^2c+3b^3c^2+c^5-\sqrt{}(1+10a^3+21a^6+8a^9+18a^4b+30a^7b+6a^2b^2+46a^5b^2+17a^8b^2+\\
&\qquad 28a^3b^3+70a^6b^3-2ab^4+51a^4b^4+8a^2b^5+66a^5b^5+22a^3b^6+4ab^7+17a^2b^8+8a^4bc+12a^7bc+2a^2b^2c+18a^5b^2c+\\
&\qquad 20a^3b^3c+22a^6b^3c+8ab^4c+66a^4b^4c+28a^2b^5c+70a^3b^6c+12ab^7c+8b^9c+4a^4bc^2+4a^7bc^2+8a^2b^2c^2+28a^5b^2c^2-\\
&\qquad 6b^3c^2+6a^3b^3c^2+51a^4b^4c^2+18a^2b^5c^2+30ab^7c^2+4a^2b^2c^3+8a^5b^2c^3+20a^3b^3c^3+4ab^4c^3+46a^2b^5c^3+4abc^4+\\
&\qquad 8a^4bc^4+8a^2b^2c^4+28a^3b^3c^4+8ab^4c^4+21b^6c^4-2c^5-6a^3c^5-2a^4bc^5+2a^2b^2c^5+18ab^4c^5+6a^2b^2c^6+10b^3c^7+c^{10}))/\\
&\quad(2(a+2a^4+2a^2b+4a^3b^2+ab^3+b^5+a^2bc+2ab^3c+a^2bc^2+b^2c^3)))\},\{1,1\}\},\\
&\{\{\tfrac{1}{2}(1+3a^3+3a^4b+3ab^4+3b^3c^2+c^5+5a^2b^2(1+c)+2ab(1+c+c^2+c^3)-\\
&\quad\sqrt{}(8a^9+17a^8b^2+8b^9c+21b^6c^4+2a^7b(15+6c+2c^2)+2a^5b^2(23+33b^3+9c+14c^2+4c^3)+(-1+c^5)^2+\\
&\qquad 2b^3c^2(-3+5c^5)+a^6(21+b^3(70+22c))+a^4b(18+8c+4c^2+8c^4-2c^5+b^3(51+66c+51c^2))+\\
&\qquad 2a^3(5-3c^5+b^6(11+35c)+b^3(14+10c+3c^2+10c^3+14c^4))+2a(2bc^4+b^7(2+6c+15c^2)+b^4(-1+4c+2c^3+4c^4+9c^5))+\\
&\qquad a^2(17b^8+2b^5(4+14c+9c^2+23c^3)+2b^2(3+c+4c^2+2c^3+4c^4+c^5+3c^6))))),0\},\\
&\{0,\tfrac{1}{2}(1+3a^3+3a^4b+3ab^4+3b^3c^2+c^5+5a^2b^2(1+c)+2ab(1+c+c^2+c^3)+\\
&\quad\sqrt{}(8a^9+17a^8b^2+8b^9c+21b^6c^4+2a^7b(15+6c+2c^2)+2a^5b^2(23+33b^3+9c+14c^2+4c^3)+(-1+c^5)^2+\\
&\qquad 2b^3c^2(-3+5c^5)+a^6(21+b^3(70+22c))+a^4b(18+8c+4c^2+8c^4-2c^5+b^3(51+66c+51c^2))+\\
&\qquad 2a^3(5-3c^5+b^6(11+35c)+b^3(14+10c+3c^2+10c^3+14c^4))+2a(2bc^4+b^7(2+6c+15c^2)+b^4(-1+4c+2c^3+4c^4+9c^5))+\\
&\qquad a^2(17b^8+2b^5(4+14c+9c^2+23c^3)+2b^2(3+c+4c^2+2c^3+4c^4+c^5+3c^6)))))\}\}\}
\end{aligned}
$$

Figure: Mathematica™ computation when iterating previous gadget

Main obstacle III

We overcome this obstacle by utilizing a novel technique of *Interpolating degenerate straddled bipartite function and splitting them into unary functions.*

Proof flowchart



Future work

We believe our dichotomy is valid even for (algebraic) real or complex-valued constraint functions. However, in this paper we can only prove it for rational-valued constraint functions.

Other immediate questions:

- Drop the $=_3$ assumption?
- Include more than one constraint function on either side?
- Other regularity parameter r?
- ...

Thank You!

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